1. Let $z$ and $w$ be complex numbers.
   (a) (4 points) Prove that $|\mathrm{Re}\, z| \leq |z|$.
   (b) (4 points) Prove that $|z+w| \leq |z|+|w|$.

2. Suppose $S$ is an ordered set with the least-upper-bound property and $E$ is a nonempty subset of $S$.
   (a) (4 points) If $\alpha = \sup E$ exists, prove that it is unique.
   (b) (4 points) If $E \subset F \subset S$ and $F$ is bounded above, prove that $E$ is bounded above and

$$\sup E \leq \sup F.$$

3. (a) (10 points) If $x, y \in \mathbb{R}$ and $x > 0$, prove that there is a positive integer $n$ such that

$$nx > y.$$

   (b) (10 points) If $x, y \in \mathbb{R}$ and $x < y$, prove that there exists a $p = \dfrac{m}{n} \in \mathbb{Q}$, $n \in \mathbb{N}$, such that

$$x < p < y \iff x < \frac{m}{n} < y \iff nx < m < ny.$$

4. (8 points) For $x, y \in \mathbb{R}$, define the function $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ by

$$d(x,y) = \frac{|x-y|}{1+|x-y|}.$$

   Prove that $d$ is a metric.

5. Let $(X,d)$ be a metric space and let $E \subset X$.
   (a) (8 points) If $p \in E$, prove that $p$ is either an interior point or a boundary point of $E$.
   (b) (8 points) If $p$ is a limit point of $E \subseteq X = (X,d)$, prove that every neighborhood of $p$ contains infinitely many points of $E$.

6. Let $X = (X,d)$ be a metric space.
   (a) (8 points) If $\{U_\alpha\}$ is a collection of open sets in $X$, prove that

$$\bigcup_\alpha U_\alpha \quad \text{is open in } X.$$

   (b) (8 points) If $K$ is a compact subset of $X$, prove that $K$ is closed.
   (c) (8 points) If $K$ is a compact subset of $X$, prove that every infinite subset $S$ of $K$ has a limit point in $K$, i.e if $S$ is a subset of $K$ containing infinitely many elements, prove that $S' \cap K \neq \emptyset$.

7. (a) (8 points) Let $E$ be a nonempty proper subset of a metric space $X$, i.e. $E \subset X$, $E \neq \emptyset$ and $E \neq X$. If $E$ is a both open and closed in $X$, prove that $X$ is disconnected.
   (b) (8 points) Let $E$ be a subset of $\mathbb{R}$. If $E$ is connected, prove that $E$ has the "interval property": if $x, y \in E$ and $x < z < y$, then $z \in E$.